

ON THE LINKING PRINCIPLE

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1. INTRODUCTION

During the last twenty years, many minimax theorems that have proved to be very useful tools in finding critical points of functionals have been established. They have all in common a geometric intersection property known as the linking principle.

Our purpose in this paper is to give a linking theorem that strengthens and unifies some of these works. We think essentially to Ambrosetti-Rabinowitz “mountain pass theorem” [1], Rabinowitz “multidimensional mountain pass theorem” [2], Rabinowitz “saddle point theorem” [3] and Silva’s variants of these results [4].

We focus our attention especially on “the limiting case”, known to be true for the mountain pass principle [5], where some information on the location of the critical points is given. We give two forms of this theorem, in the first part of the paper, the first one is established via a deformation lemma and in the second part we use Ekeland’s variational principle to get the second one. Unfortunately, we could not remove the finite dimension condition that appears in the results [5] and [6]. This finite dimension assumption is dropped only when dealing with a special kind of functionals having a predefined shape [7, 8, 4].

2. USING THE DEFORMATION LEMMA

We start by giving a short description of the notion of linking that seems to be more natural than the initial one introduced by Benci and Rabinowitz. It formalizes a topological property of intersection that appears in all the results cited above.

Definition 1. Let S be a closed subset of a Banach space V , Q is a subset of V with relative boundary ∂Q , we say S and ∂Q link if :

$$(L1) \quad S \cap \partial Q = \emptyset$$

$$(L2) \quad \text{for all } \gamma \in \mathcal{C}(V, V) \text{ such that } \gamma|_{\partial Q} = I_d \text{ we have } \gamma(Q) \cap S \neq \emptyset$$

More generally, if Γ is a subset of $\mathcal{C}(V, V)$, then S and ∂Q are said to link with respect to Γ if the relation (L1) holds and (L2) is satisfied for any $\gamma \in \Gamma$.

The following examples yield respectively the geometry of the saddle point theorem and the generalized mountain pass theorem.

Example 1. Let $V = V_1 \oplus V_2$ be a space decomposed into two closed subspaces V_1 and V_2 with $\dim V_2 < \infty$.

Let $S = V_1$ and $Q = B_R(0) \cap V_2$ with relative boundary

$$\partial Q = \{u \in V_2; \|u\| = R\}.$$

Then S and ∂Q link.

Example 2. Let $V = V_1 \oplus V_2$ as in Example 1 and let $e \in V_1$ with $\|e\| = \rho$ be given. Suppose $0 < \rho < R_1, 0 < R_2$ and let

$$S = \{u \in V_1; \|u\| = \rho\},$$

$$Q = \{se + u_2; 0 \leq s \leq R_1, u_2 \in V_2 \text{ and } \|u_2\| \leq R_2\}$$

with relative boundary

$$\partial Q = \{se + u_2; s \in \{0, R_1\} \text{ or } \|u_2\| = R_2\}.$$

Then S and ∂Q link.

These examples are now well known in the specialized literature (see for instance [7]). For a proof see [9] or [10]. A short hint is given in the Appendix.

In minimax theorems, the functional must verify some compactness property known as the Palais-Smale condition. We recall it.

Definition 2. Let $\Phi \in \mathcal{C}^1(V, \mathbb{R})$ and $c \in \mathbb{R}$. We say that Φ satisfies the Palais-Smale Condition (P.S.) if the existence of a sequence $(u_n)_n$ in V such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ implies that $(u_n)_n$ possesses a convergent subsequence. And Φ satisfies the local Palais-Smale condition at c denoted by $(P.S.)_c$, if the existence of a sequence $(u_n)_n$ in V such that $\Phi(u_n) \rightarrow c$ and $\Phi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ implies that it is precompact.

2.1. A Generalized linking theorem. We will use in the sequel the following notations:

$$\Phi^a = \{v \in V; \Phi(v) \leq a\}, \quad \Phi_b^a = \{v \in V; b \leq \Phi(v) \leq a\},$$

$$\mathcal{K}_c = \{v \in V; \Phi'(v) = 0 \text{ and } \Phi(v) = c\},$$

$$N_\delta(E) = \{v \in V; \text{dist}(v, E) = \|v - E\| \leq \delta\},$$

and

$$\tilde{V} = \{v \in V; \Phi'(v) \neq 0\}.$$

Let us now state the abstract critical point theorem we announced in the beginning.

Theorem 1. Suppose that $\Phi \in \mathcal{C}^1(V, \mathbb{R})$, $S \subset V$ is a closed subset and $Q \subset V$ satisfy:

- (a) S and ∂Q link with respect to Γ .
- (b) There exists $\alpha \in \mathbb{R}$ such that

$$\Phi|_{\partial Q} \leq \alpha \leq \Phi|_S$$

- (c) There exists $\gamma_0 \in \mathcal{C}(V, V)$ such that

$$\sup_{u \in Q} \Phi(\gamma_0(u)) < \infty.$$

Let

$$c_\Gamma = \inf_{\gamma \in \Gamma} \sup_{u \in Q} \Phi(\gamma(u)),$$

if Φ satisfies (P.S.)_{c_Γ} then c_Γ defines a critical value and $c_\Gamma \geq \alpha$.

Moreover if $c_\Gamma = \alpha$ then $\mathcal{K}_{c_\Gamma} \cap S \neq \emptyset$.

By Γ we denote one of the sets

- $\Gamma_1 = \{\gamma \in \mathcal{C}(V, V); \gamma|_{\partial Q} = I_d\}$
- $\Gamma_2 = \{\gamma \in \mathcal{K}(V, V): \text{set of compact maps}; \gamma|_{\partial Q} = I_d\}$
- $\Gamma_3 = \{\gamma \in \mathcal{H}(V, V): \text{set of homeomorphisms}; \gamma|_{\partial Q} = I_d\}$
- $\Gamma_4 = \{\gamma \in \mathcal{K}(V, V): \gamma(\text{closed set}) = \text{closed set and } \gamma|_{\partial Q} = I_d\}$
- $\Gamma_5 = \{\gamma \in \Gamma_1 \text{ such that } \gamma(\text{closed bounded set}) = \text{compact set}\}$

Compact means continuous and maps bounded sets into relatively compact ones.

Remark 1. • The assumption (c) is satisfied when Q is compact.

- In the classical results, Q is compact and in general it is assumed that there exists $\beta < \alpha$ such that

$$\Phi|_{\partial Q} \leq \beta < \alpha \leq \Phi|_S.$$

- If we denote by \mathcal{F} the set of the Γ_i defined above, it is still possible to consider for Γ any finite intersection in \mathcal{F} .

Remark 2. The set used in general in the classical results is Γ_1 . The set Γ_3 has also been used, but Γ_2 , Γ_4 and Γ_5 have never been used at our best knowledge. They have a particular importance because the sets in examples 1 and 2 link with respect to Γ_2 , Γ_4 and Γ_5 too, *even when $\dim V_2$ is infinite*¹. Unfortunately, if these sets are small enough to make that ∂Q and S link, they are empty when $\dim V_2$ is infinite (See Remark 10 at the end).

Notice also that we do not require that the values of Φ on ∂Q and S are *strictly separated*.

The reader is supposed familiarized with these classical results. So even improved, we will not state them here.

¹This could lead to removing the finite dimension assumption of V_2

We want to point out also that

$$\begin{cases} c_{\Gamma_3} \geq c_{\Gamma_1} \geq \alpha, \text{ and} \\ c_{\Gamma_5} \geq c_{\Gamma_4} \geq c_{\Gamma_2} \geq c_{\Gamma_1} \geq \alpha \end{cases} \quad \text{where the } c_{\Gamma_i} \text{'s are defined in Theorem 1}$$

If $\Gamma_i \subset \Gamma_j$ then $c_{\Gamma_j} \geq c_{\Gamma_i} \geq c_{\Gamma_1} \geq \alpha$ and there is no reason they must be equal in general. Nevertheless this has not been verified.

The proof follows a classical method. We use a slight modification of the standard deformation lemma. It can be found in [8] when $V = V_1 \oplus V_2$ is a Hilbert space and $\Phi(u) = (Lu, u) + b(x)$ with $L = L_1P_1 + L_2P_2$ where $P_i: V \rightarrow V_i$ is the projection onto V_i , the operator $L_i: V_i \rightarrow V_i$ is bounded selfadjoint and the derivative b' of b is compact.

Lemma 2. *Let V a Banach space, D and E two closed subsets of V and $f: V \rightarrow \mathbb{R}$ a \mathcal{C}^1 -functional satisfying (P.S.)_c for $c \in \mathbb{R}$ such that*

(L1) $D \cap E = \emptyset$ and $E \cap \mathcal{K}_c = \emptyset$, and

(L2) $f|_E \leq c \leq f|_D$.

Then, there exist $\varepsilon > 0$ and $\eta \in \mathcal{C}(\mathbb{R} \times V, V)$ such that

- (i) $\eta_t = \eta(t, \cdot): V \rightarrow V$ is a homeomorphism and $\eta_t^{-1} = \eta_{-t}$, for all t .
- (ii) $f(\eta(t, x)) \leq f(x)$, for all $t \geq 0$,
- (iii) $\eta(t, x) = x$, for all $t \in \mathbb{R}$, for any $x \in D$, and
- (iv) $f(\eta(1, x)) \leq c - \varepsilon$, for all $x \in E$

There are many deformation lemmas in the literature, some were proved sometimes with weaker conditions than (P.S.) (see for example [4, 10]). The proof of this one is practically given in [8], we will sketch it in the Appendix for the convenience of the reader.

Proof. Proof of Theorem 1 • Notice that by assumption (c), $c_\Gamma < \infty$.

- since S and ∂Q link, clearly $c_\Gamma \geq \alpha$. So c_Γ is well defined.

We will distinguish the two cases:

- **Suppose that $c_\Gamma > \alpha$.** Let us suppose that $\mathcal{K}_{c_\Gamma} = \emptyset$. Set $\bar{\varepsilon} = (c_\Gamma - \alpha)/2$. By the classical deformation lemma (see [6]) we have that for any $\varepsilon < \bar{\varepsilon}$, there exists $\eta \in \mathcal{C}([0, 1] \times V, V)$ such that

$$\eta(1, \Phi^{c_\Gamma + \varepsilon}) \subset \Phi^{c_\Gamma - \varepsilon}.$$

But by the definition of c_Γ , there exists $\gamma \in \Gamma$ such that $\sup_{u \in Q} \Phi(\gamma(u)) < c_\Gamma + \varepsilon$. Setting $\gamma' = \eta(1, \cdot) \circ \gamma$ we have $\sup_{u \in Q} \Phi(\gamma'(u)) < c_\Gamma - \varepsilon$. This implies that $\gamma'|_{\partial Q} = I_a$, so $\gamma' \in \Gamma$ but this contradicts the definition of c_Γ .

- **Suppose $c_\Gamma = \alpha$.** We claim that $\mathcal{K}_{c_\Gamma} \cap S \neq \emptyset$. Indeed, if by contradiction this was not the case. Then, since

$$-\Phi|_S \leq -c_\Gamma \leq -\Phi|_{\partial Q}$$

By Lemma 2.1, there exists ε and $\eta \in \mathcal{C}(\mathbb{R} \times V, V)$ such that $\eta(t, x) = x$ on ∂Q and $-\Phi(\eta(1, x)) \leq -c_\Gamma - \varepsilon$ for all x in S . But the definition of c_Γ implies

that there exists $\gamma \in \Gamma$ such that

$$\Phi(\gamma(x)) < c_\Gamma + \varepsilon \quad \text{for all } x \in Q. \quad (*)$$

Let $\gamma'(x) = \eta_1^{-1}(\gamma(x))$, then by (iii) and the choice of Γ_i , the functional $\gamma' \in \Gamma$. Now, since S and ∂Q link, there exists $\bar{x} \in Q$ such that $\gamma'(\bar{x}) \in S$ and then $\Phi(\gamma(\bar{x})) = \Phi(\eta(1, \gamma'(\bar{x}))) \geq c_\Gamma + \varepsilon$ and this contradicts $(*)$. \square

In [4], Silva has shown with an other deformation lemma some variants of the classical results cited above, which, at least at a first look, don't seem to be a part of the general linking principle stated here. We mention a saddle point theorem with some *global estimates*.

Theorem 3. *Let $V = V_1 \oplus V_2$ be a real Banach space with $\dim V_2 < +\infty$, $\Phi \in \mathcal{C}^1(V, \mathbb{R})$ such that*

$$(a) \quad \Phi|_{V_2} \leq \beta, \quad \beta \in \mathbb{R}.$$

And

$$(b) \quad \Phi|_{V_1} \geq \alpha, \quad \alpha \in \mathbb{R}.$$

If Φ verifies (P.S.)_c, for any $c \in [\alpha, \beta]$, then Φ admits a critical value $c_\Gamma \in [\alpha, \beta]$, characterized by a minimax argument.

This result can also be obtained with Theorem 2.1 even if it uses global estimates instead of local ones in which the linking sets are explicited. Indeed, by Silva's deformation lemma, there exist ε , $R_0 > 0$ and $\eta \in \mathcal{C}([0, 1] \times V, V)$ such that

$$\eta(1, u) \in \Phi^{\alpha-\varepsilon}, \quad \text{for all } u \in \Phi^\beta \setminus \mathbb{B}(0, R_0)$$

Since by Example 1, $S = V_1$ and ∂Q link when $Q = \mathbb{B}(0, R_0) \cap V_2$. We will be in the situation of Theorem 1 if we prove that $c = c_\Gamma \in [\alpha, \beta]$. And this last statement is true. Indeed, since ∂Q and S link. For all $\gamma \in \Gamma$, we have $\gamma(Q) \cap S \neq \emptyset$, so $\sup_Q \Phi \circ \gamma \geq \inf_S \Phi \geq \alpha$ and hence

$$c_\Gamma = \inf_{\gamma \in \Gamma} \sup_Q \Phi \circ \gamma \geq \alpha.$$

The fact that the restriction $\gamma|_{\partial Q} = I_{d_{V_2}}$ implies that $P_2 \circ \gamma|_{\partial Q} = I_{d_{V_2}}$ where $P_2: V \rightarrow V_2$ is the projection onto V_2 . But $\gamma' = P_2 \circ \gamma \in \Gamma$ and $\gamma'(Q) = P_2 \circ \gamma(Q) \subset V_2$. Therefore $\sup_Q \Phi \circ \gamma' \leq \beta$ and $c_\Gamma = \inf_{\gamma \in \Gamma} \sup_Q \Phi \circ \gamma \in [\alpha, \beta]$.

We mention equally an other abstract critical point theorem by Silva:

Theorem 4 (Silva). *Let $V = V_1 \oplus \mathbb{R}e \oplus V_2$ be a real Banach space with $\dim V_2 < \infty$, where $\|e\| = 1$ and $\Phi \in \mathcal{C}^1(V, \mathbb{R})$ are such that:*

$$(a) \quad \text{There exists } \rho > 0 \text{ and } \alpha \geq 0 \text{ such that } \Phi|_{S_\rho} \geq \alpha.$$

where $S_\rho = \overline{(V_1 \setminus \mathbb{B}(0, \rho)) \cup ((\mathbb{R}^+ e \oplus V_1) \cap \partial \mathbb{B}(0, \rho))}$.

$$(b) \quad \Phi|_{V_2 \oplus \mathbb{R}e} \leq \beta.$$

If Φ satisfies $(P.S.)_c$ for all c in $[\alpha, \beta]$, then Φ possesses a critical value c_r in $[\alpha, \beta]$ characterized by a minimax argument.

Remark 3. We point out that in Theorem 7 we have $\alpha \leq \Phi(0) \leq \beta$ and $\alpha \leq \Phi(\rho e) \leq \beta$ in Theorem 4.

In this last result, Silva's deformation lemma implies that (see [4]): There exists $R_0 > \rho$ a real $\varepsilon \in]0, \alpha[$ and a deformation $\eta \in \mathcal{C}([0, 1] \times V, V)$ such that $\eta(1, u) \in \Phi^{\alpha-\varepsilon}$ for all $u \in (\Phi^\beta \setminus \mathbb{B}(0, R_0)) \cap (V_2 \oplus \mathbb{R}e)$. But if we set $Q = \mathbb{B}(0, R_0) \cap (V_2 \oplus \mathbb{R}e)$ then S_ρ , as defined in Theorem 3, and ∂Q link. We give the proof in the Appendix. There too, Theorem 1 applies if we prove that $c_r \in [\alpha, \beta]$. And this is proved similarly to what we did in the proof of Theorem 3.

Remark 4. In all the results cited here, no finite dimension is required to prove the linking with respect to the particular sets Γ_i in use.

Remark 5. The theorems seen here are of course applied in finding stationary points of functionals and then to solve variational problems. But unfortunately they have the drawback to impose the compactness condition $(P.S.)_c$ which is unlikely to verify and seems rather restrictive. But recently Struwe mentioned in [9] that the failure of $(P.S.)_c$ at certain levels reflects, using physicists terminology, phenomena related to “phase transition” or “particle creation” at these levels, (for more details see [9]).

3. USING EKELAND'S VARIATIONAL PRINCIPLE

Now, we will use some convex analysis results and Ekeland's variational principle to prove an other variant of the abstract theorem stated in the first part. The fact that the critical point is located on S in the limiting case, will be confirmed again.

3.1. Ekeland's variational principle. When a functional is l.s.c. (lower semi-continuous) in a reflexive Banach space, it possesses a minimum if and only if it has a bounded minimizing sequence. But if the space is not reflexive or the functional is only lower semi-continuous, we can say no thing similar. Ekeland has introduced in 1972 a “variational principle” that applies well in such situations.

Theorem 5 (Ekeland's variational principle). *Let (X, d) be a complete metric space and $\Phi: X \rightarrow \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous functional, bounded from below and not identical to $+\infty$. Then, for all $\varepsilon > 0$, each $\delta > 0$ and each $x \in X$ such that*

$$\Phi(x) \leq \inf_{x \in X} \Phi(x) + \varepsilon.$$

There exists $y \in X$ with the properties

- a) $\Phi(y) \leq \Phi(x)$
- b) $\text{dist}(x, y) \leq \delta$
- c) $\Phi(z) > \Phi(y) - \frac{\varepsilon}{\delta} \text{dist}(z, y)$ for all $z \neq y$ in X

This principle has been successfully used many times to prove the existence of “almost critical points” of unbounded functionals via a minimax method.

Theorem 6. *Let X be a Banach space, K a compact metric space, $K_0 \subset K$ a closed subset, $\mu \in \mathcal{C}(K_0, X)$ and $E = E(K, K_0, X, \mu)$ the complete metric space defined by*

$$E = \{m \in \mathcal{C}(K, X); m|_{K_0} = \mu\}$$

endowed with the distance of uniform convergence on K .

Let $f \in \mathcal{C}^1(X, \mathbb{R})$ and set

$$c = \inf_{m \in E} \max_{s \in K} f(m(s))$$

and

$$d = \max_{s \in K_0} f(\mu(s)).$$

Suppose that

$$c > d.$$

Then, for each $\varepsilon \in]0, c - d[$ and each $p \in E$ such that

$$\max_{s \in K} f(p(s)) \leq c + \varepsilon$$

there exists $u \in X$ such that

$$c - \varepsilon \leq f(u) \leq \max_{s \in K} f(p(s)),$$

$$\text{dist}(u, p(K)) \leq \sqrt{\varepsilon}$$

and

$$\|f'(u)\| \leq \sqrt{\varepsilon}.$$

In [11], [12] and [13], two nice proofs of this result using Ekeland’s variational principle are given.

This theorem contains as particular cases the theorems cited in the first part. Unfortunately, in this theorem the notion of “linking” described before is hidden, and the strict separation of the values of the functional on ∂Q and S is essential in the proofs.²

We will use Ekeland’s principle to prove an abstract critical point theorem close to Theorem 1 where we both exhibit the notion of linking and treat the “limiting case”. We want to point out that, in this part, in addition to [11, 12, 13] we have been inspired by the paper of [14] devoted to the particular case of the mountain pass theorem.

²In fact, the linking is reintroduced again by the authors to obtain the particular cases cited above.

Theorem 7. *Let X be a Banach space, Q a compact subset and S a closed subset of X such that S and ∂Q link.*

Let $f: X \rightarrow \mathbb{R}$ be a \mathcal{C}^1 -functional and set

$$E = \{\gamma \in \mathcal{C}(Q, X); \gamma|_{\partial Q} = I_d\},$$

$$c = \inf_{\gamma \in E} \max_{x \in Q} f(\gamma(x))$$

and

$$c_0 = \inf_S f, \text{ and } d = \max_{\partial Q} f$$

Suppose that

$$c_0 \geq d. \quad (*)$$

Then f admits a sequence of “almost critical” points $(u_n)_n \subset E$ such that $f(u_n) \rightarrow c$ and $f'(u_n) \rightarrow 0$. In the case $c = c_0$, (in this case $c = c_0 = d$) we have some informations on the location of these points. For each $0 < \varepsilon < \max\{1, \text{dist}(\partial Q, S)\}/2$, there exists $x_\varepsilon \in X$ such that:

- (i) $c \leq f(x_\varepsilon) \leq c + \frac{5}{4}\varepsilon^2$,
- (ii) $\text{dist}(x_\varepsilon, S) \leq \frac{3}{2}\varepsilon$, and
- (iii) $\|f'(x_\varepsilon)\| \leq \frac{\varepsilon}{2}$.

PROOF. Since S and ∂Q link, the relation $(*)$ implies that

$$c \geq c_0 \geq d.$$

If $c > c_0$ or $c_0 > d$ then $c > d$ and we are in the situation of Theorem 6. So, it suffices to treat the limiting case $c = c_0 = d$. In fact, instead of $(*)$ we will use only the condition $c = c_0$, but in this case it is legitimate to wonder if we have not the situation

$$d > c_0.$$

The answer is negative, we have really $c_0 \geq d$ and therefore $(*)$. Indeed, if there exists $t \in \partial Q$ with $f(t) > c_0$. Since $t \in \partial Q$, we would have $\gamma(t) = t$ for each $\gamma \in E$ and hence $f(\gamma(t)) = f(t) > c_0$ for each $\gamma \in E$.

Therefore

$$c \geq f(t) \geq c_0,$$

which contradicts the equality $c = c_0$.

Before continuing the proof, we will recall some known results uses in the sequel and whose proofs can be found in standard convex analysis literature.

Preliminaries. Let $\Phi: X \rightarrow \mathbb{R}$ be a functional, the subdifferential of Φ is the multifunction $\partial\Phi: X \rightarrow \mathcal{P}(X^*) = 2^{X^*}$ defined by

$$\partial\Phi(x) = \{\mu \in X^*; \Phi(y) \geq \Phi(x) + \langle \mu, y - x \rangle, \forall y \in X\}$$

where X^* is the dual space of X .

Proposition 8. *Let $\Phi: X \rightarrow \mathbb{R}$ be continuous and convex. Then for each $x, y \in X$*

$$\lim_{t \downarrow 0} \frac{\Phi(x + ty) - \Phi(x)}{t} = \max_{\mu \in \partial Q(x)} \langle \mu, y \rangle.$$

Proposition 9. *Let $\Phi: X \rightarrow \mathbb{R}$ be continuous and convex. Then $\partial\Phi(x)$ is non-empty, convex and w^* -compact in X^* for each $x \in X$.*

Let K be a compact metric space and $\mathcal{E} = \mathcal{C}(K, \mathbb{R})$ the Banach space of continuous functions for the norm $\|x\| = \max\{x(t); t \in K\}$. By Riesz representation Theorem, the dual space \mathcal{E}^* of \mathcal{E} is isometric isomorphic to the Banach space of regular Radon measures defined on the σ -algebra of Borel sets of K .

Recall that:

- A Radon measure μ is positive ($\mu \geq 0$) if $\langle \mu, x \rangle \geq 0$ for all $x \in \mathcal{E}$ such that $x(t) \geq 0$ for each $t \in K$.
- A Radon measure has mass one if $\langle \mu, \mathbb{1} \rangle = 1$, where $\mathbb{1}: K \rightarrow \mathbb{R}$ is the constant function $\mathbb{1}(t) = 1$ for any $t \in K$.
- A Radon measure vanishes in an open subset $U \subset K$ if $\langle \mu, x \rangle = 0$ for each $x \in \mathcal{E}$ such that the support of x is a compact set $K \subset U$. If μ vanishes in a collection of open subsets $(U_\alpha)_\alpha$, then μ vanishes in $\cup_\alpha U_\alpha$, therefore there exists a largest open set \tilde{U} where it vanishes. The support of μ , denoted by $\text{supp } \mu$, is defined by $\text{supp } \mu = K \setminus \tilde{U}$.

Proposition 10. *Let $\theta: \rightarrow \mathbb{R}$, $x \mapsto \theta(x) = \max\{x(t); t \in K\}$. Then θ is convex and continuous. Moreover, for each $x \in E$*

$$\mu \in \partial\theta(x) \iff \begin{cases} \mu \geq 0, \langle \mu, \mathbb{1} \rangle = 1, \\ \text{supp } \mu \subset \{t \in K; x(t) = \theta(x)\} \end{cases}$$

THE PROOF OF THEOREM 7 CONTINUED:

We recall that we are in the limiting case $c = c_0$.

Let $g \in E$ such that

$$(1) \quad \max_{t \in Q} f(g(t)) < c + \frac{\varepsilon^2}{4}$$

Let us denote by

$$S_{g,\varepsilon} = \{x \in X; \text{dist}(g(x), S) < \varepsilon\}$$

and set

$$A = Q \cap S_{g,\varepsilon} = \{x \in Q; \text{dist}(g(x), S) < \varepsilon\}$$

A is non-empty because by the linking of S and ∂Q , $g(Q) \cap S \neq \emptyset$.

Set

$$\Gamma(A) = \{m \in \mathcal{C}(\bar{A}, X); m = g \text{ on } \partial A\}$$

The closure \bar{A} of A is compact and X is a Banach space. Hence $\Gamma(A)$ endowed with the distance

$$\text{dist}_A(k_1, k_2) = \max_{x \in A} \|k_1(x), k_2(x)\|$$

is a complete metric space.

Set

$$\Psi(x) = \max \{0, \varepsilon^2 - \varepsilon \cdot \text{dist}(x, S)\}$$

and

$$\mathcal{J}: \Gamma(A) \rightarrow \mathbb{R}, \quad k \mapsto \mathcal{J}(k) = \max_{\bar{A}} \{f(k(t)) + \Psi(f(t))\}$$

- For each function $k \in \Gamma(A)$, we have

$$k(\bar{A}) \cap S \neq \emptyset.$$

Indeed, if we note by $\tilde{k} = k \cdot \chi_{\bar{A}} + g \cdot \chi_{A^c}$, it is obvious that $\tilde{k} \in E$ because $k = g$ on ∂A . Since S and ∂Q link, there exists $\bar{t} \in Q$ such that $\tilde{k}(\bar{t}) \in S$. But by the definition of A , $\text{dist}(g(t), S) \geq \varepsilon$ on $A^c \cap Q$, therefore $\bar{t} \in \bar{A}$.

- Using the relation (*), we have³

$$\mathcal{J}(k) \geq f(k(\bar{t})) + \Psi(k(\bar{t})) \geq c + \varepsilon^2$$

Since k was taken arbitrary in $\Gamma(A)$,

$$(2) \quad \inf_{\Gamma(A)} \mathcal{J} \geq c + \varepsilon^2.$$

But if we set $\tilde{g} = g|_{\bar{A}}$, we have

$$(3) \quad \mathcal{J}(\tilde{g}) = \max_{t \in \bar{A}} \{f(g(t)) + \Psi(g(t))\},$$

$$(4) \quad \leq \max_{t \in Q} \{f(g(t)) + \Psi(g(t))\}.$$

Therefore

$$(5) \quad \mathcal{J}(\tilde{g}) \leq \left(c + \frac{\varepsilon^2}{4}\right) + \varepsilon^2 = c + \frac{5\varepsilon^2}{4}.$$

Since \mathcal{J} is bounded from below by $c + \varepsilon^2$ and lower semi-continuous in the complete metric space $\Gamma(A)$ and $\tilde{g} \in \Gamma(A)$ verifies

$$\mathcal{J}(\tilde{g}) \leq \inf_{\Gamma(A)} \mathcal{J} + \frac{\varepsilon^2}{4}.$$

Ekeland's variational principle implies then that there exists $\hat{g} \in \Gamma(A)$ such that

$$(6) \quad \mathcal{J}(\hat{g}) \leq \mathcal{J}(\tilde{g}),$$

$$(7) \quad \|\hat{g} - \tilde{g}\| \leq \frac{\varepsilon}{2}$$

and

$$(8) \quad \mathcal{J}(k) \geq \mathcal{J}(\hat{g}) - \frac{\varepsilon}{2} \|k - \hat{g}\|, \quad \forall k \in \Gamma(A)$$

- Let M be the subset of \bar{A} where $f \circ \hat{g}$ realizes its maximum in \bar{A} .

CLAIM: *There exists $t_0 \in M$ such that:*

$$\|f'(\hat{g}(t_0))\| \leq \frac{3}{2}\varepsilon.$$

³Since $c = c_0 = \inf_S f$ and $k(\bar{t}) \in S$.

Indeed, for any $h \in \mathcal{C}(\bar{A}, X)$ such that $h|_{\partial A} \equiv 0$, we have

$$f(\hat{g}(t_0) + \lambda h(t)) = f(\hat{g}(t)) + \lambda \langle f'(g(t)), h(t) \rangle + o(\lambda h(t))$$

Therefore

$$\max_{\bar{A}} f(\hat{g}(t_0) + \lambda h(t)) \leq \max_{t \in \bar{A}} \{f(\hat{g}(t)) + \lambda \langle f'(g(t)), h(t) \rangle\} + o(\lambda \|h\|)$$

where $\|h\| = \max_{\bar{A}} \|h(t)\|$.

Using this relation in (8), we obtain

$$\mathcal{J}(g_\varepsilon) \geq \mathcal{J}(\hat{g}) + \frac{\varepsilon}{2} \|g_\varepsilon - \hat{g}\| \quad \text{where } g_\varepsilon = \hat{g} + \lambda h.$$

Therefore

$$\max_{\bar{A}} f(\hat{g}(t)) \leq \max_{t \in \bar{A}} \{f(\hat{g}(t)) + r\lambda \langle f'(g(t)), h(t) \rangle\} + \varepsilon \|h\|/2$$

So denoting

$$\alpha(t) = f(\hat{g}(t)), \quad \beta(t) = \langle f'(t), h(t) \rangle$$

and

$$N(\gamma) = \max_{\bar{A}} \gamma \quad \text{with } \gamma \in \mathcal{C}(\bar{A}, X).$$

We have

$$N(\alpha + \lambda\beta) - N(\alpha)/\lambda \geq -\varepsilon \|h\|/2, \quad \forall \lambda > 0$$

Therefore

$$(9) \quad \liminf_{\lambda \downarrow 0} [N(\alpha + \lambda\beta) - N(\alpha)]/\lambda \geq -\varepsilon \|h\|/2.$$

Remark 6. Notice that β depends on h and that $M \cap Q = \emptyset$.

Set $N(\gamma)$ the subdifferential of N at γ . We recall that

$$\partial N(\gamma) = \{\mu; \mu \text{ is a Radon measure of mass 1 with support in } M(\gamma)\}$$

Using Proposition 3 in [9], we have:

$$\begin{aligned} -\frac{\varepsilon}{2} \|h\| &\leq \liminf_{\lambda \downarrow 0} [N(\alpha + \lambda\beta) - N(\alpha)] \\ &\leq \max \{ \langle \beta, \mu \rangle / \mu \in \partial N(\alpha) \} \\ &= \max \left\{ \int_{\bar{A}} \langle f'(\hat{g}), h \rangle d\mu; \mu \in \partial N(\alpha) \right\} \end{aligned}$$

Now, apply [15, Theorem 6.2.7] to the functional:

$$\mathcal{G}: \mathcal{M}(\bar{A}, \mathbb{R}) \times \mathcal{C}(A, X) \rightarrow \mathbb{R}$$

$$\mathcal{G}(\mu, k) = \langle \mu, \langle f'(\hat{g}(.)), k(t) \rangle \rangle = \int_{\bar{A}} \langle f'(\hat{g}), h \rangle d\mu$$

Let $\mathcal{M}(K, \mathbb{R})$ be the Banach space of Radon measures defined on the σ -Algebra of all Borel sets of \bar{A} endowed with the w^* -topology. Then \mathcal{G} is continuous, linear

in each variable separately. And the sets $\partial N(\alpha)$ and $\{k \in \mathcal{C}(K, X); \|k\| \leq 1\}$ are convex, the former one being w^* -compact. We have

$$\begin{aligned} -\frac{\varepsilon}{2}\|h\| &\leq \inf_h \max_\mu \left\{ \int_{\bar{A}} \langle f'(\hat{g}), h \rangle d\mu; \mu \in \partial N(\alpha), \|h\| \leq 1, h|_{\partial \bar{A}} = 0 \right\} \\ &= \max_\mu \inf_h \left\{ \int_{\bar{A}} \langle f'(\hat{g}), h \rangle d\mu; \mu \in \partial N(\alpha), \|h\| \leq 1, h|_{\partial \bar{A}} = 0 \right\} \\ &= \max_\mu \left\{ \int_{\bar{A}} \|f'(\hat{g})\| d\mu; \mu \in \partial N(\alpha) \right\} \\ &= -\min\{\|f'(\hat{g}(t))\|; t \in \underbrace{M(f'(\hat{g}))}_\alpha\} \end{aligned}$$

Therefore, there exists $t_0 \in M$ such that $\|f'(\hat{g}(t_0))\| \leq \frac{3}{2}\varepsilon$ and the Claim is proved.

For (i): By (4), (5), (6) and the Claim, we have

$$c + \varepsilon^2 \leq \inf_{\Gamma(A)} \mathcal{J} \leq f(\hat{g}(t_0) + \Psi(\hat{g}(t_0))) = \mathcal{J}(\hat{g}) \leq \mathcal{J}(\tilde{g}) \leq c + \frac{5}{4}\varepsilon^2.$$

Since $0 \leq \Psi \leq \varepsilon^2$, we have $c \leq f(x_\varepsilon) \leq c + \frac{5}{4}\varepsilon^2$.

For (ii): It suffices to see that since $t_0 \in \bar{A} = \{x \in Q; \text{dist}(g(x), S) \leq \varepsilon\}$, we have $\text{dist}(\tilde{g}(t_0), S) = \text{dist}(g(t_0), S) \leq \varepsilon$. Then by (7), we have

$$\text{dist}(x_\varepsilon, S) = \text{dist}(\hat{g}(t_0), S) \leq \frac{3}{2}$$

□

Remark 7. Assuming that f verifies $(P.S.)_c$, the inf max value c is critical in the limiting case and there exists a critical point of level c in S . This confirms the result obtained by the deformation lemma.

Remark 8. In [11], Mawhin has pointed out that in such situations we need a weaker condition than $(P.S.)_c$ to conclude. Indeed, since we are sure that there exists a sequence (u_k) such that:

$$f(u_k) \rightarrow c \quad \text{and} \quad f'(u_k) \rightarrow 0$$

It suffices then to assume that f is such that the existence of such a sequence implies that c is a critical value.

Unfortunately, in applications, c isn't known explicitly. This constrains us to verify $(P.S.)_c$ for all possible values c .

As corollaries of the abstract results, we obtain the known theorems:

Corollary 11 (Rabinowitz [16]). *Let $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ a functional that satisfies $(P.S.)$. Suppose that*

$$\inf_{u \in B(0, r)} \Phi(u) = \max\{\Phi(0), \Phi(e)\} = c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \Phi(\gamma(t))$$

where $0 < r < \|e\|$ and $\Gamma = \{\gamma \in \mathcal{C}([0, 1], X); \gamma(0) = 0 \text{ and } \gamma(1) = e\}$
Then Φ has a critical point of level c on the sphere $S(0, r)$.

Corollary 12 (Pucci-Serrin [17]). *Let $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ satisfying (P.S.), if Φ has a pair of local minima (or local maxima) then Φ possesses a third critical point.*

APPENDIX

In the proof of the linking in Examples 1 and 2, we use the homotopy property of the degree theory respectively with:

$$F_t: Q \rightarrow V_2, \quad \text{where} \quad F_t(u) = tP_2\gamma(u) + (1-t)u \quad t \in [0, 1]$$

and to

$$F_t: Q \rightarrow \overline{\mathbb{R} \times V_2}, \quad \text{such that} \quad \text{for all } u \in V_2 \cap B_{R_2} \quad \text{and for all } s \in [0, R_1]$$

$$F_t(u) = (||tP_1\gamma(u + se)|| - \rho + (1-t)s, tP_2\gamma(u + se) + (1-t)u)$$

It is obvious there that the finite dimension of V_2 can be dropped if we require γ to be compact and the degree will still have a sense as Schauder's degree of a compact perturbation of the identity.

Proof of the linking of Q_R and S_ρ in Theorem 4.

We want to prove it without assuming $\dim V_2 < \infty$. When assuming $\dim V_2 \leq \infty$ the proof becomes easier. We will adapt the proof of Silva to our situation. The compactness of the elements of $\Gamma_4, \Gamma_5, \Gamma_2, \Gamma_2 \cap \Gamma_3$ compensating the local compactness of the finite dimensional space and allowing the use of degree theory as explained above.

The space $V = V_1 \oplus \mathbb{R}e \oplus V_2$, while the reals ρ and R are such that $0 < \rho < R$ $D_R = \partial B_R(0) \cap (V_1 \oplus \mathbb{R}e)$ and $S_\rho = \overline{V_2 \setminus B_\rho(0)} \cup (\partial B_\rho(0) \cap (\mathbb{R}^+e \oplus V_2))$,

If we take $\Gamma = \Gamma_4$ or $\Gamma = \Gamma_5$ or $\Gamma = \Gamma_2 \cap \Gamma_3$ then S_ρ and D_R link with respect to Γ . Indeed:

- First, remark that $S_\rho \cap \partial D_R = \emptyset$ because if $x \in S_\rho \cap \partial D_R$, then we would have

$$\left\{ \begin{array}{l} ||x|| = R \\ x \in V_1 \oplus \mathbb{R}e \\ x \in V_2 \end{array} \right\} \Rightarrow \begin{array}{l} ||x|| = R, \\ x = 0. \end{array}$$

And this yields a contradiction.

- We claim that for all $\gamma \in \Gamma$, it holds that $\gamma(D_R) \cap S_\rho \neq \emptyset$. Indeed, let

$$\chi_\beta = \begin{cases} 1 & \text{if } x \geq \rho/\beta \\ \beta x/\rho & \text{if } x \in [0, \rho/\beta] \\ 0 & \text{if } x \leq 0 \end{cases}$$

Notice that $\chi_\beta \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, let P_1 be the projection $P_1: V \rightarrow V_1$ and $P_e: V \rightarrow \mathbb{R}e$ the projection onto $\mathbb{R}e$. Set

$$G_{\beta,s}(re + v) = \left(s(re + v) + (1-s) \left[P_1 \left(\gamma(re + v) \right) + \left[\chi_\beta \left(P_e \left(\gamma(re + v) \right) \right) \left\| (I_d - P_1)(\gamma(re + v)) \right\| \right] e \right] \right) \quad (1)$$

where $re + v$ denotes a generic element of $V_1 \oplus \mathbb{R}e$ and $s \in [0, 1]$, then $G_{\beta,s} \in \mathcal{C}(V_1 \oplus \mathbb{R}e, V_1 \oplus \mathbb{R}e)$.

If $re + v \in \partial D_R$, since $\gamma|_{\partial D_R} = I_d$, we have

$$G_{\beta,s}(re + v) = \left(sr + v + (1-s)\chi_\beta(r)|r| \right) \quad (2)$$

Therefore, if $\beta > 1$ we have that $\rho e \notin G_{\beta,s}(\partial D_R)$ for any $s \in [0, 1]$ because if $G_{\beta,s}(re + v) = \rho e$, then $r = R$ and $v = 0$, and this implies that $G_{\beta,s}(re + v) = Re$, and this contradicts the fact that $\rho < R$.

So, the topological degree of the compact perturbation of the identity $G_{\beta,s}$ on D_R at ρe is well defined. By the homotopy invariance of the degree, we have

$$d(G_{\beta,s}, D_R, \rho e) = d(I_d, D_R, \rho e) = 1,$$

Therefore, for $s = 0$ we conclude that

$$\begin{cases} P_1 \gamma(u_\beta) = 0, \\ \chi_\beta(P_e \gamma(u_\beta)) \cdot \|(I_d - P_1)\gamma(u_\beta)\| = \rho > 0. \end{cases} \quad (3)$$

By the definition of χ_β and the second equation of (3), we have that $P_e \gamma(u_\beta) > 0$. Then, if we suppose that $\gamma(D_R) \cap S_\rho = \emptyset$ we would have $P_e(\gamma(u_\beta)) < \rho/\beta$ where $\beta > 1$. If that wasn't the case, $\chi_\beta(P_e \gamma(u_\beta)) = 1$ and hence $\|(I_d - P_1)\gamma(u_\beta)\| = \rho$. But since $P_1(\gamma(u_\beta)) = 0$, we would obtain $\|\gamma(u_\beta)\| = \rho$ and $\gamma(u_\beta) \in V_2 \oplus \mathbb{R}e$, i.e. $\gamma(u_\beta) \in S_\rho$? A contradiction with what we supposed before, so $P_e \gamma(u_\beta) < \rho/\beta$.

Let $(\beta_m)_m \subset \mathbb{R}$ a sequence such that $\beta_m \rightarrow \infty$ as $m \rightarrow \infty$ and $(u_{\beta_m})_m \subset D_R$ the sequence given by (3) and satisfying

$$0 < P_e(\gamma(u_{\beta_n})) < \rho/\beta_n \quad (4)$$

Since D_R is bounded and $(u_{\beta_n})_n \subset D_R$, the sequence $(\gamma(u_{\beta_n}))_n$ is relatively compact. So, for a subsequence still denoted by $(u_{\beta_n})_n$ we have $\gamma(u_{\beta_n}) \rightarrow u \in V_2 \oplus \mathbb{R}e$ because $\gamma \in \Gamma$ is compact.

By (4) $P_e u = \lim_n P_e \gamma(u_{\beta_n}) = 0$ and since $P_1 \gamma(u_{\beta_n}) = 0$ for any $n \in \mathbb{N}$ we have

$$P_e u = P_1 u = 0.$$

But $\chi_\beta \leq 1$, so by the second equation of (3) we have

$$\|u\| = \|(I_d - P_1)u\| = \lim_n \|(I_d - P_1)\gamma(u_{\beta_n})\| \geq \rho$$

and hence $u \in \overline{(V_2 \setminus B_\rho(0))} \subset S_\rho$ i.e. $u \in \overline{\gamma(D_R)} \cap S_\rho$. While by the definition of Γ taken here we have $\gamma(D_R) = \gamma(D_R)$. A contradiction with $\gamma(D_R) \cap S_\rho = \emptyset$. \square

Remark 9. We saw that the finite dimension of V_2 isn't required to prove the linking in any of the cases presented here when we use the smaller sets of "compact" functions, so why can't we use them to remove this additional assumption? The answer is that these sets are so small that they are empty. At least this happens with Hilbert spaces, indeed, let $H = V_1 \oplus V_2$ with both $\dim V_1$ and $\dim V_2$ infinite, and suppose by contradiction that the set

$$\Gamma = \{\gamma \in \mathcal{C}(H, H) \text{ compact; } \gamma|_{S(0,R) \cap V_2} = I_d\}$$

is not empty. Without loss of generality, we can suppose $R = 1$ so that $S(0, R)$ is the unit sphere. Let us take an orthonormal basis $(u_n)_n$ of V_2 , we know that it converges weakly to 0. Take $\gamma \in \Gamma$, since the unit sphere is bounded and γ compact, $\gamma(S(0, R))$ is compact and the sequence $\gamma(u_n) = u_n$ admits a subsequence converging to an element of $S(0, R)$, a contradiction with the fact that $u_n \rightharpoonup 0$.

Proof of Lemma 2

The details will be omitted since the proof is not difficult and classical but quite long. It suffices to change f' by a pseudogradient field in the proof of [8]. We sketch it here in few lines for the convenience of the reader.

The map η is obtained as a solution of a differential equation. First we easily verify that

$$(1) \overline{N_\delta(E)} \cap \mathcal{K}_c = \emptyset, \text{ and}$$

$$(2) \|f'(x)\| \geq b, \text{ for all } x \in A = N_\delta(E) \cap f_{c-\bar{\varepsilon}}^{c+\bar{\varepsilon}}.$$

Set $A_1 = V \setminus (\overline{N_{\delta/2}(E)} \cap f_{c-\bar{\varepsilon}/2}^{c+\bar{\varepsilon}/2})$, and $A_2 = N_{\delta/3}(E) \cap f_{c-\bar{\varepsilon}/3}^{c+\bar{\varepsilon}/3}$ and define

$$h(x) = \|x - A_1\| / (\|x - A_1\| + \|x - A_2\|),$$

$$\rho(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 1 \\ 1/s & \text{if } s \geq 1 \end{cases}$$

$$X_1(x) = \begin{cases} -\delta/3 \ h(x) \rho(\|W(x)\|) W(x) & \text{in } A \\ 0 & \text{in } V \setminus A \end{cases}$$

where W is a pseudogradient field defined on \tilde{V} . And let ζ be the solution of

$$\begin{cases} d\zeta/dt &= X_1(\zeta) \\ \zeta(0, x) &= x \in V \end{cases}$$

Then, if $E_1 = \zeta([0, 1] \times E)$, we prove easily using the graph norm that E_1 is closed and with some obvious properties of ζ that $E_1 \cap D = \emptyset$.

Let $D_1 = \{x \in V; d(x) \leq 1\}$, where $d(x) = \|x - D\| / (\|x - D\| + \|x - E_1\|)$. It is clear then that $D_1 \cap E_1 = \emptyset$.

Define now $X(x) = q(x)X_1(x)$ with $q(x) = \|x - D_1\| / (\|x - D_1\| + \|x - E_1\|)$ then

the solution η of

$$\begin{cases} d\eta/dt &= X(\eta) \\ \eta(0, x) &= x \in V \end{cases}$$

satisfies the desired properties (the proof there is standard except for some obvious changes in (iv)).

Note. Definition 1 has its origin in [9]. After a first version of this paper was written, it was brought to our attention that the same definition appears also in [18]. In Willem's paper an interesting result near to our Theorem 7 is given [18, Lemme 2.18]:

Theorem 13 (Théorème de localisation de Willem). *Let X be a Banach space, $\Phi \in \mathcal{C}^1(X, \mathbb{R})$, $F, Q \subset X$. Write*

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in Q} \Phi(\gamma(u))$$

where

$$\Gamma = \{\gamma \in \mathcal{C}(Q, X); \gamma(u) = u \text{ on } \partial Q\}$$

If

(D1) F and ∂Q link,

(D2) $\text{dist}(F, \partial Q) > 0$,

(D3) $-\infty < c = \inf_F \Phi$,

Then for each $\varepsilon > 0$, $\delta \in]0, \text{dist}(F, \partial Q)/2[$ and $\gamma \in \Gamma$ such that

$$\sup_Q \Phi \circ \gamma < c + \varepsilon$$

There exists $u \in X$ such that

a) $c - 2\varepsilon \leq \Phi(u) \leq c + 2\varepsilon$

b) $\text{dist}(u, F \cap \gamma(Q)_\delta) \leq 2\delta$

c) $\|\Phi'(u)\| < 4\varepsilon/\delta$.

We report some remarks

- In Willem's result, Q isn't supposed compact so the assumption $-\infty < c$ which appears in (D3) is necessary.
- When Q is compact, our result contains Theorem 13. Our result contains also classical point theorems cited above while this isn't the case for 13.
- The proof of our Theorem 7 uses "Ekeland's Variational principle" while Willem uses a *General Deformation Lemma* to prove his result.

Acknowledgment—We are also grateful to Professor M. Willem for sending us his reference [18].

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